The Derivatives of Two Types of Functions

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Abstract

This paper takes the mathematical software Maple as the auxiliary tool to study the differential problem of two types of functions. We can obtain the Fourier series expansions of any order derivatives of these functions by using differentiation term by term theorem and Leibniz differential rule, and hence greatly reduce the difficulty of calculating their higher order derivative values. In addition, we provide two functions to evaluate their any order derivatives, and calculate some higher order derivative values practically. The research methods adopted in this study involved finding solutions through manual calculations and verifying these solutions by using Maple. This type of research method not only allows the discovery of calculation errors, but also helps modify the original directions of thinking from manual and Maple calculations. For this reason, Maple provides insights and guidance regarding problem-solving methods.

Keywords: derivatives; Fourier series expansions; differentiation term by term theorem; Leibniz differential rule; Maple

1. Introduction

The computer algebra system (CAS) has been widely employed in mathematical and scientific studies. The rapid computations and the visually appealing graphical interface of the program render creative research possible. Maple possesses significance among mathematical calculation systems and can be considered a leading tool in the CAS field. The superiority of Maple lies in its simple instructions and ease of use, which enable beginners to learn the operating techniques in a short period. In addition, through the numerical and symbolic computations performed by Maple, the logic of thinking can be converted into a series of instructions. The computation results of Maple can be used to modify our previous thinking directions, thereby forming direct and constructive feedback that can aid in improving understanding of problems and cultivating research interests. Inquiring through an online support system provided by Maple or browsing the Maple website (www.maplesoft.com) can facilitate further understanding of Maple and might provide unexpected insights. As for the instructions and operations of Maple, we can refer to [1]-[7].

In calculus courses, finding the n-th order derivative value $f^{(n)}(c)$ of a function $f(x)$ at $x = c$, in general, needs to go through two procedures: firstly evaluating the n-th order derivative $f^{(n)}(x)$ of $f(x)$, and secondly taking $x = c$ into $f^{(n)}(x)$. These two procedures will make us face with increasingly complex calculations when calculating higher order derivative values of a function (i.e. n is large), Therefore, to obtain the answers by manual calculations is not easy. In this paper, we study the differential problems of the following two types of functions

$$f(x) = e^{ax} \ln[b + c\cos(\lambda x + \beta)] \tag{1}$$

$$g(x) = e^{ax} \ln[b + c\sin(\lambda x + \beta)] \tag{2}$$

where a, b, c, λ, β are real numbers, $\lambda \neq 0$, $b > 0$. We can obtain the Fourier series expansions of any order derivatives of these two types of functions by using differentiation term by term theorem and Leibniz differential rule, and hence greatly reduce the difficulty of evaluating their higher order derivatives values ; these are the major results of this paper (i.e., Theorems 1, 2). As for the study of related differential problems can refer to [8]-[21]. In addition, we propose two examples to do calculation practically. The research methods adopted in this study involved finding solutions through manual calculations and verifying these solutions by using Maple. This type of research method not only allows the discovery of calculation errors, but also helps modify the original directions of thinking from manual and Maple calculations. For this reason, Maple provides insights and guidance regarding problem-solving methods.

2. Main Results

Firstly, we introduce a notation and two formulas used in this study.

Notation.

Let $z = a + ib$ be a complex number, where $i = \sqrt{-1}$, a, b are real numbers. We denote a the real part of z by $\mathrm{Re}(z)$, and b the imaginary part of z by $\mathrm{Im}(z)$.

Formulas.

(i) $\displaystyle\sum_{k=1}^{\infty} \frac{1}{k} (-x)^k = -\ln(1+x)$, where $-1 < x \leq 1$.

(ii) geometric series. $\displaystyle\frac{1}{1+z} = \sum_{k=0}^{\infty} (-1)^k z^k$, where z is a complex number, $|z| < 1$.

Next, we introduce two important theorems used in this paper.

Differentiation term by term theorem ([22]).

If, for all non-negative integer k, the functions $g_k : (a,b) \to R$ satisfy the following three conditions : (i) there exists a point $x_0 \in (a,b)$ such that $\displaystyle\sum_{k=0}^{\infty} g_k(x_0)$ is convergent, (ii) all functions $g_k(x)$ are differentiable on open interval (a,b), (iii) $\displaystyle\sum_{k=0}^{\infty} \frac{d}{dx} g_k(x)$ is uniformly convergent on (a,b). Then $\displaystyle\sum_{k=0}^{\infty} g_k(x)$ is uniformly convergent and differentiable on (a,b). Moreover, its derivative $\displaystyle\frac{d}{dx} \sum_{k=0}^{\infty} g_k(x) = \sum_{k=0}^{\infty} \frac{d}{dx} g_k(x)$.

Leibniz differential rule ([23]). Let n be a positive integer. If $f(x), g(x)$ are functions such that their m-th derivatives $f^{(m)}(x), g^{(m)}(x)$ exist for all $m = 1,..., n$. Then the n-th derivative of product function $f(x)g(x)$,

$$(fg)^{(n)}(x) = \sum_{m=0}^{n} \binom{n}{m} f^{(n-m)}(x) g^{(m)}(x)$$

, where $\displaystyle\binom{n}{m} = \frac{n!}{m!(n-m)!}$.

Before deriving our major results, we need a lemma.

Lemma A. Let b, c, λ, β be real numbers, $\lambda \neq 0, b > |c|$. The Fourier series expansion of $\ln[b + c\cos(\lambda x + \beta)]$,

$$\ln[b + c\cos(\lambda x + \beta)] = 2\ln\left(\frac{\sqrt{b+c} + \sqrt{b-c}}{2}\right) - 2\sum_{k=1}^{\infty}\frac{1}{k}\left(-\frac{\sqrt{b+c} - \sqrt{b-c}}{\sqrt{b+c} + \sqrt{b-c}}\right)^k \cos(\lambda kx + \beta k)$$

(3)

for all $x \in R$.

Proof. Let $p = \frac{1}{2}(\sqrt{b+c} + \sqrt{b-c}), q = \frac{1}{2}(\sqrt{b+c} - \sqrt{b-c})$, then the derivative of $\ln[b + c\cos(\lambda x + \beta)]$,

$$\frac{d}{dx}\ln[b + c\cos(\lambda x + \beta)]$$

$$= \frac{-c\lambda\sin(\lambda x + \beta)}{b + c\cos(\lambda x + \beta)}$$

$$= \frac{-c\lambda\sin(\lambda x + \beta)}{p^2 + 2pq\cos(\lambda x + \beta) + q^2}$$

$$= \frac{\dfrac{-c\lambda\sin(\lambda x + \beta)}{q^2}}{\left(\dfrac{p}{q} + \cos(\lambda x + \beta)\right)^2 + \sin^2(\lambda x + \beta)}$$

$$= \operatorname{Im}\left[\frac{\dfrac{c\lambda}{q^2}\left(\dfrac{p}{q} + \cos(\lambda x + \beta) - i\sin(\lambda x + \beta)\right)}{\left(\dfrac{p}{q} + \cos(\lambda x + \beta) + i\sin(\lambda x + \beta)\right)\left(\dfrac{p}{q} + \cos(\lambda x + \beta) - i\sin(\lambda x + \beta)\right)}\right]$$

$$= \frac{c\lambda}{pq}\cdot\operatorname{Im}\left(\frac{1}{1 + \dfrac{qz}{p}}\right) \quad \text{(where } z = e^{i(\lambda x + \beta)}\text{)}$$

$$= 2\lambda\operatorname{Im}\left[\sum_{k=0}^{\infty}\left(-\frac{qz}{p}\right)^k\right] \quad \left(\text{because }\left|-\frac{qz}{p}\right| = \left|\frac{q}{p}\right| < 1 \text{, we can use geometric series}\right)$$

$$= 2\lambda\sum_{k=1}^{\infty}\left(-\frac{q}{p}\right)^k \sin(\lambda kx + \beta k)$$

(4)

Therefore,

$$\ln[b + c\cos(\lambda x + \beta)] = -2\sum_{k=1}^{\infty}\frac{1}{k}\left(-\frac{q}{p}\right)^k \cos(\lambda kx + \beta k) + K$$

(5)

where K is a constant.

Taking $x = -\dfrac{\beta}{\lambda}$ into both sides of (5), we obtain

$$\ln(b+c) = -2 \sum_{k=1}^{\infty} \frac{1}{k}\left(-\frac{q}{p}\right)^k + K \tag{6}$$

It follows that

$$K = \ln(b+c) + 2 \sum_{k=1}^{\infty} \frac{1}{k}\left(-\frac{q}{p}\right)^k$$

$$= \ln(b+c) - 2\ln\left(1 + \frac{q}{p}\right) \quad \text{(by formula (i))}$$

$$= 2\ln p \tag{7}$$

Hence,

$$\ln[b + c\cos(\lambda x + \beta)]$$

$$= 2\ln p - 2 \sum_{k=1}^{\infty} \frac{1}{k}\left(-\frac{q}{p}\right)^k \cos(\lambda k x + \beta k)$$

$$= 2\ln\left(\frac{\sqrt{b+c} + \sqrt{b-c}}{2}\right) - 2 \sum_{k=1}^{\infty} \frac{1}{k}\left(-\frac{\sqrt{b+c} - \sqrt{b-c}}{\sqrt{b+c} + \sqrt{b-c}}\right)^k \cos(\lambda k x + \beta k) \qquad \blacksquare$$

The following is the first major result in this study, we determine the Fourier series expansions of any order derivatives of function (1).

Theorem 1. Suppose a, b, c, λ, β are real numbers, $\lambda \neq 0$, $b > |c|$, n is any positive integer, and let the domain of $f(x) = e^{ax}\ln[b + c\cos(\lambda x + \beta)]$ be $(-\infty, \infty)$. The n-th order derivative of $f(x)$,

$$f^{(n)}(x) = -2e^{ax} \sum_{m=0}^{n} \binom{n}{m} a^{n-m} \cdot \lambda^m \sum_{k=1}^{\infty} k^{m-1}\left(-\frac{\sqrt{b+c} - \sqrt{b-c}}{\sqrt{b+c} + \sqrt{b-c}}\right)^k \cos\left(\lambda k x + \beta k + \frac{m\pi}{2}\right)$$

$$+ 2a^n \ln\left(\frac{\sqrt{b+c} + \sqrt{b-c}}{2}\right) e^{ax} \tag{8}$$

for all $x \in R$.

Proof. $\quad f^{(n)}(x)$

$$= \frac{d^n}{dx^n} e^{ax}\ln[b + c\cos(\lambda x + \beta)]$$

$$= \sum_{m=0}^{n} \binom{n}{m} (e^{ax})^{(n-m)} (\ln[b + c\cos(\lambda x + \beta)])^{(m)} \qquad \text{(by Leibniz differential rule)}$$

$$= \sum_{m=1}^{n} \binom{n}{m} a^{n-m} \cdot e^{ax} \left[-2\lambda^m \cdot \sum_{k=1}^{\infty} k^{m-1} \left(-\frac{\sqrt{b+c} - \sqrt{b-c}}{\sqrt{b+c} + \sqrt{b-c}} \right)^k \cos\left(\lambda kx + \beta k + \frac{m\pi}{2} \right) \right]$$

$$+ a^n e^{ax} \left[2\ln\left(\frac{\sqrt{b+c} + \sqrt{b-c}}{2} \right) - 2\sum_{k=1}^{\infty} \frac{1}{k} \left(-\frac{\sqrt{b+c} - \sqrt{b-c}}{\sqrt{b+c} + \sqrt{b-c}} \right)^k \cos(\lambda kx + \beta k) \right]$$

(by Lemma A and differentiation term by term theorem)

$$= -2e^{ax} \sum_{m=0}^{n} \binom{n}{m} a^{n-m} \cdot \lambda^m \sum_{k=1}^{\infty} k^{m-1} \left(-\frac{\sqrt{b+c} - \sqrt{b-c}}{\sqrt{b+c} + \sqrt{b-c}} \right)^k \cos\left(\lambda kx + \beta k + \frac{m\pi}{2} \right)$$

$$+ 2a^n \ln\left(\frac{\sqrt{b+c} + \sqrt{b-c}}{2} \right) e^{ax}$$

for all $x \in R$ ■

In Theorem 1, if replacing β by $\beta - \dfrac{\pi}{2}$, then we immediately obtain the Fourier series expansions of any order derivatives of function (2).

Theorem 2. If the assumptions are the same as Theorem 1, and the domain of $g(x) = e^{ax} \ln[b + c\sin(\lambda x + \beta)]$ is $(-\infty, \infty)$. The n-th order derivative of $g(x)$,

$$g^{(n)}(x) = -2e^{ax} \sum_{m=0}^{n} \binom{n}{m} a^{n-m} \cdot \lambda^m \sum_{k=1}^{\infty} k^{m-1} \left(-\frac{\sqrt{b+c} - \sqrt{b-c}}{\sqrt{b+c} + \sqrt{b-c}} \right)^k \cos\left(\lambda kx + \beta k + \frac{(m-k)\pi}{2} \right)$$

$$+ 2a^n \ln\left(\frac{\sqrt{b+c} + \sqrt{b-c}}{2} \right) e^{ax} \qquad (9)$$

for all $x \in R$.

3. Examples

Next, aimed at the differential problem of the two types of functions in this study, we provide two functions and use Theorems 1, 2 to determine the Fourier series expansions of their any order derivatives and evaluate some of their higher order derivative values practically. In addition, we use Maple to calculate the approximations of these higher order derivative values and their solutions for verifying our answers.

Example 1. Suppose the domain of the function

$$f(x) = e^{2x} \ln\left[5 - 4\cos\left(3x - \frac{5\pi}{6} \right) \right] \qquad (10)$$

is $(-\infty, \infty)$.

Then by Theorem 1, we obtain the n -th order derivative of $f(x)$,

$$f^{(n)}(x) = -2e^{2x} \sum_{m=0}^{n} \binom{n}{m} 2^{n-m} \cdot 3^{m} \sum_{k=1}^{\infty} k^{m-1} \left(\frac{1}{2}\right)^{k} \cos\left(3kx - \frac{5k\pi}{6} + \frac{m\pi}{2}\right) + 2^{n+1} \ln 2 \cdot e^{2x} \qquad (11)$$

for all $x \in R$.

Therefore, we obtain the 12-th order derivative value of $f(x)$ at $x = \frac{\pi}{2}$,

$$f^{(12)}\left(\frac{\pi}{2}\right) = -2e^{\pi} \sum_{m=0}^{12} \binom{12}{m} 2^{12-m} \cdot 3^{m} \sum_{k=1}^{\infty} k^{m-1} \left(\frac{1}{2}\right)^{k} \cos\left(\frac{2k\pi}{3} + \frac{m\pi}{2}\right) + 2^{13} \ln 2 \cdot e^{\pi} \qquad (12)$$

In the following, we use Maple to verify the correctness of (12).

```
>f:=x->exp(2*x)*ln(5-4*cos(3*x-5*Pi/6));
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$$f := x \rightarrow e^{2x} \ln\left(5 - 4\cos\left(3x - \frac{5}{6}\pi\right)\right)$$

```
>evalf((D@@12)(f)(Pi/2),24);
```

$$5.914923419957997989107 \cdot 10^{9}$$

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>evalf(-2*exp(Pi)*sum(12!/(m!*(12-m)!)*2^(12-m)*3^m*sum(k^(m-1)*(1/2)^k*cos(2*k*Pi/3+m*Pi/2),k=1
..infinity),m=0..12)+2^13*ln(2)*exp(Pi),22);
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$$5.914923419957997989129 \cdot 10^{9}$$

Example 2. Let the domain of the function

$$g(x) = e^{-3x} \ln\left[10 + 8\sin\left(4x + \frac{2\pi}{3}\right)\right] \qquad (13)$$

be $(-\infty, \infty)$.

By Theorem 2, we can evaluate the n -th order derivative of $g(x)$,

$$g^{(n)}(x) = -2e^{-3x} \sum_{m=0}^{n} \binom{n}{m}(-3)^{n-m} \cdot 4^{m} \sum_{k=1}^{\infty} k^{m-1} \left(-\frac{1}{2}\right)^{k} \cos\left(4kx + \frac{k\pi}{6} + \frac{m\pi}{2}\right) + 2(-3)^{n} \cdot \ln(2\sqrt{2}) \cdot e^{-3x}$$

$$(14)$$

for all $x \in R$.

Therefore, we obtain the 9-th order derivative value of $g(x)$ at $x = -\frac{5\pi}{6}$,

$$g^{(9)}\left(-\frac{5\pi}{6}\right)$$

$$= -2e^{5\pi/2} \sum_{m=0}^{9} \binom{9}{m}(-3)^{9-m} \cdot 4^{m} \sum_{k=1}^{\infty} k^{m-1} \left(-\frac{1}{2}\right)^{k} \cos\left(\frac{5k\pi}{6} + \frac{m\pi}{2}\right) + 2(-3)^{9} \cdot \ln(2\sqrt{2}) \cdot e^{5\pi/2} \qquad (15)$$

Using Maple to verify the correctness of (15) as follows:

>g:=x->exp(-3*x)*ln(10+8*sin(4*x+2*Pi/3));

$$g := x \rightarrow e^{-3x} \ln\left(10 + 8\sin\left(4x + \frac{2}{3}\pi\right)\right)$$

>evalf((D@@9)(g)(-5*Pi/6),24);

$$1.07193719414228756252776 \cdot 10^{14}$$

>evalf(-2*exp(5*Pi/2)*sum(9!/(m!*(9-m)!)*(-3)^(9-m)*4^m*sum(k^(m-1)*(-1/2)^k*cos(5*k*Pi/6+m*Pi/2),

k=1..infinity),m=0..9)+2*(-3)^9*ln(2*sqrt(2))*exp(5*Pi/2),24);

$$1.07193719414228756252766 \cdot 10^{14} - 0.\,I$$

The above answer obtained by Maple appears I $(= \sqrt{-1})$, it is because Maple calculates by using special functions built in. The imaginary part is zero, so can be ignored.

4. Conclusion

As mentioned, the differentiation term by term theorem and the Leibniz differential rule play significant roles in the theoretical inferences of this study. In fact, the applications of these two theorems are extensive, and can be used to easily solve many difficult problems; we endeavor to conduct further studies on related applications. On the other hand, Maple also plays a vital assistive role in problem-solving, we can even use Maple to design some types of differential problems, and try to find the methods to solve them. In the future, we will extend the research topic to other calculus and engineering mathematics problems and solve these problems by using Maple. These results will be used as teaching materials for Maple on education and research to enhance the connotations of calculus and engineering mathematics.

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